Accessibility between Entangled States

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Abstract

Entanglement, a weird quantum correlation between separate physical systems, is an indispensable resource for quantum information processing. As with other physical resources, the key to understanding its full potential is to quantify it. In this paper, I review our recent insights into the quantification of entanglement based on the operational notion of accessibility between physical states. In particular, I show that the partially ordered structure of mixed entangled states underlies the non-uniqueness of their entanglement measure.

1. Introduction

Correlation is an essential ingredient of information processing. Indeed, the very purpose of communication is to create a correlation between a sender and a receiver so that they share the same information after the transmission. Private cryptographic keys shared between two parties are correlations that enable secure communication. Also, computation is a process that correlates input with output. All of these are correlations of classical information, which is operated according to classical physics.

Quantum theory, which describes the physics of the microscopic world, offers yet another type of correlation called *entanglement*. Entanglement is a weird link between two (or more) quantum systems in that it cannot be explained in a classical manner. However, it is this weirdness that makes entanglement an indispensable resource for quantum information processing. In quantum teleportation [1], entanglement acts as a sort of channel. In quantum cryptography [2], it can change into cryptographic keys. It is also believed to be the key to efficient quantum computation [3].

With the resource of entanglement, we have entered a new era of information processing that is far more powerful than anything conceived before. However, we still have a long way to go before we grasp the fundamental laws governing the behavior of entanglement. As with other physical resources like energy, quantification is the key to understanding the full potential of entanglement. We need to establish a measure to specify the amount of entanglement in order to use it effectively and efficiently. In this paper, we explore whether it is possible to define a measure of entanglement uniquely.

For bipartite pure-state entanglement, a unique measure has already been established, which is known as the entropy of entanglement. Interestingly, it turns out that thermodynamics and the theory of bipartite pure-state entanglement share the same mathematical structure. The proof of the uniqueness of the entanglement measure is therefore essentially the same as that of the uniqueness of entropy in quantifying thermal equilibrium states [4]. However, for mixed-state entanglement, several different measures have been proposed; there is not a unique one because different measures are needed for quantifying mixed-state entanglement in different scenarios [5]. In this paper, I show that the non-uniqueness of the entanglement measure originates in the structure of the entire set of entangled states, which is classified according to the operational notion of *accessibility* between those states.

Entanglement is in principle a general notion applicable to any quantum system. It can thus exist between a pair of photons, a pair of atoms, and so on. Therefore, the following argument holds in any rep-

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resentation of qubits (two-level quantum systems), although its exposition might look rather abstract. This is a very important feature of quantum information theory in general, which "distills" only properties intrinsic to quantum theory itself and is independent of specific systems. In other words, how well we can manipulate information quantumly depends heavily on how well we understand the structure of Hilbert space operationally. (For an introduction to quantum information theory, see Ref. [6].)

This paper is organized as follows. Section 2 introduces the basic ideas of entanglement manipulation: the definition of entanglement and the framework in which we deal with it. Section 3 revisits, from a very general point of view, the way we quantify a physical quantity. The relationship between the accessibility of physical states and their ordering is presented. Section 4 reviews the unique measure for bipartite pure-state entanglement. Section 5 discusses the main result of this paper: the non-uniqueness of the entanglement measure and the accessibility between entangled states. Finally, section 6 summarizes the paper.

2. Entanglement manipulation

This section defines entanglement and presents the framework that we use for dealing with it, which is the minimal prerequisite for understanding the rest of the paper.

Suppose two parties, say Alice and Bob, are distantly located and share the following two-qubit quantum state called a Bell pair, with each qubit being possessed by one party:

$$\frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right). \tag{1}$$

We say a state is *entangled* when the state of the entire system cannot be written as a tensor product of the states of each part, like Eq. (1). Conversely, if a state can be written in a product form like $|0\rangle_A \otimes |0\rangle_B$, then it is not entangled. Indeed, the above example of a Bell pair cannot be factorized into a product form.

This is the definition of pure entangled states. The entire state, the two-qubit state in the above example, is a pure quantum state, for which one can have complete knowledge about the identity of the state. However, there is another type of quantum states called mixed states, for which all one can know is a statistical distribution over possible quantum states. For example, suppose a quantum system is in a state $|\psi\rangle$ with probability p and in a state $|\phi\rangle$ with probability $1-p$. The state of the system is written as $\rho = p|\psi\rangle\langle\psi| + (1-p)|\phi\rangle\langle\phi|$. This is an example of mixed states, which are described by density matrices. In other words, pure states are special cases where one can know the identity of the state of the system with probability 1.

The definition of entangled states can be extended to mixed states as well. A composite quantum system is called entangled when the state cannot be written as a statistical mixture of product states

$$\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}, \tag{2}$$

where $\{p_i\}$ is the probability distribution, and $\rho_A^{(i)}$ and $\rho_B^{(i)}$ are local density matrices for Alice and Bob, respectively. This is a natural generalization of the definition of pure entangled states.

Note that it is crucial in the definition of entangled states that one recognizes that the entire system is composed of two (or more) parts so that one can speak about the correlation between the parts. If the entire system is considered to be *one* quantum system located in *one* place (for example, the Bell state in Eq. (1) can be viewed as one state of a four-level system), then the notion of entanglement disappears. Entanglement relies on the tensor-product structure of Hilbert space, which is expected to have a rich mathematical structure and deep implications for the power of quantum information processing.

In order to fully appreciate the properties of entanglement and the tensor-product structure of Hilbert space, we deal with it in a framework that is slightly different from that used in traditional physics. As was already pointed out, it is crucial that entangled systems are distributed at different places. We therefore restrict ourselves to a scenario in which the two parties are allowed to access only their own systems locally and they can talk over a classical communication channel. In local operations, each party can manipulate his/her own system at will by local unitary time evolution and local quantum measurement and can also add local ancillary quantum systems to the original system, if necessary. A classical communication channel, say a traditional telephone line, is necessary because distant parties might need to tell each other their local measurement outcomes. This is the framework of "local operations and classical communication" (LOCC), on which our discussion of the properties of entanglement is based.

In ordinary discussions of entanglement, we usually assume that entangled states are already shared between distant parties somehow and see what will emerge under the LOCC framework. In other words, the study of entanglement mainly focuses on investigating the behavior of composite quantum systems under LOCC. A simple example of LOCC protocols is that of quantum teleportation [1]. Teleportation allows two distant parties, Alice and Bob, to communicate a quantum state, without them actually sending a quantum system itself, just by local quantum operations and classical communication, provided that they shared a Bell pair in advance. In fact, they need to send one qubit in the Bell state through a quantum communication channel in advance. This means that they do need a quantum channel in this sense. One Bell pair is needed for transmission of one qubit because the correlation in the Bell pair is broken and thus consumed after teleportation. However, for any input state, one type of entanglement, a Bell pair, is enough to accomplish teleportation successfully, thus Alice and Bob can prepare Bell pairs much earlier than actual transmission of the input quantum state, even before they have made up their mind to perform the communication! This is the reason entangled states work as quantum communication channels and thereby serve as a resource for quantum information processing.

Now that we have seen an aspect of entanglement as a resource, a natural question arises: What if Alice and Bob share different types of entangled states? There are many different types of entangled states besides Bell states. States can be partially entangled in some situations, or might even be degraded into some mixed entangled states during the initial transmission of one half of a Bell pair, resulting in a less entangled state. Therefore, to use the resource of entanglement efficiently, we must develop a theory for quantifying different types of entangled states, hopefully in a unique way.

3. Accessibility between physical states

In this section, before moving on to the question of whether we can quantify entanglement uniquely, we revisit from a general viewpoint the way in which physical quantities are quantified. This approach to quantification is based on the notion of *accessibility*, which enables us to make the argument fully operational.

Accessibility between two physical states by some physical operation is crucial in order to compare the states quantitatively. When there exists an *operation* that converts one state into the other, we say that the latter is accessible from the former via the operational process. Then, we can derive an ordering between the two states from the accessibility based on the operation. This ordering between two states (together with a few other natural assumptions) makes it possible to define a quantity that compares them. The fact that the ordering is naturally derived from some physical time evolution (operation) means that the quantification is not just artificial but physically reasonable. However, if there is no operation that converts one state into the other in either direction within a given framework, it is impossible to conceive any coherent way to compare them. One can allot any number to the states to quantify them artificially, but such quantification has no physical significance.

The uniqueness of a measure to quantify a certain physical property relies heavily on the ordering properties of the entire state space. When all elements in a given set of physical states can be completely ordered, i.e., any two states are ordered, we say the set is *totally ordered* (**Fig. 1(a)**). We can then make at least one consistent measure that quantifies the states in the set; one can align all the states based on the accessibility and induce a measure according to the alignment. Note that this property itself does not guarantee the uniqueness of the measure. On the other hand, if there is no ordering that works globally in the set, i.e., a certain pair of states cannot be ordered, we say that the set is *partially ordered* (**Fig. 1(b)**). We fail to find a consistent way to "align" all the states, which implies that there is no unique measure.

As seen in the above, total order itself is not a suffi-

Fig. 1. (a) Total order: State X and state Y are accessible at least in one direction.
(b) Partial order: State X and state Y are not accessible from each other in either direction.

cient condition for the uniqueness of a measure but a necessary condition for it. Any set of physical states that can be quantified uniquely must have a totally ordered structure based on a relevant operation. The contraposition of this statement means that if a state space is partially ordered, then the set cannot be quantified uniquely. This plays an important role in the following argument for the non-uniqueness of the entanglement measure.

Let us take a look at some familiar examples of total and partial order to get a feeling for these abstract definitions. These concepts occur in ordinary physics, where they play fundamental roles, although they might not always be recognized explicitly.

The most beautiful and successful application of the theory of ordering physical states is in thermodynamics, where totally ordered structure appears as a basis of the uniqueness of entropy. Consider the set of all thermal equilibrium states and the adiabatic operations between them. Here, by adiabatic processes we mean processes in which thermodynamical systems, say gas in a cylinder, are operated without exchanging heat with the environment. We do not require that the processes be quasi-static, i.e., infinitesimally slow. States may change at any rate; for example, the piston in the cylinder may move rapidly. It turns out that the set of the equilibrium states is totally ordered under adiabatic processes. From the totally ordered structure (together with a few natural assumptions), we can derive that entropy is the unique measure to quantify the set of equilibrium states: Given two equilibrium states, A and B, entropy S distinguishes between possible and impossible directions of the adiabatic processes between them. A can access B via an adiabatic process if and only if $S(A) \leq S(B)$. If the equality holds, B can also access A, so the process is reversible. Quite interestingly, this property of thermodynamics parallels the theory of bipartite pure entanglement, in which entropy is also the unique measure of entanglement that distinguishes between possible and impossible directions of entanglement manipulation. We will briefly review this connection in the next section. (For the operational approach to thermodynamics mentioned here, see Ref. [4] and the references contained therein.)

One of the most familiar examples in physics that contains partial order is in the special theory of relativity. Consider a pair of events whose light cones include each other, i.e., the interval between the two events is time-like. Then, the events are accessible because one can affect the other by sending a signal. However, if one is outside the light cone of the other, i.e., the interval between the two events is space-like, then it is impossible to connect them by any physical operation because nothing can travel faster than light. Consequently, there is no unique way of ordering two such events; different ordering is possible by choosing different reference frames. Therefore, the set of events in space time is a partially ordered one, which leads to the well-known non-uniqueness of simultaneity that follows from the principles of special theory of relativity. We will see a similar structure in mixed entangled states below.

4. Total order on bipartite pure entangled states

This section briefly reviews the well-established theory of bipartite pure-state entanglement from the viewpoint of ordering states. It will be seen that the uniqueness of the entanglement measure for bipartite pure states, the entropy of entanglement, can be understood on the same basis as that of the operational approach to thermodynamics mentioned above.

Suppose Alice and Bob share the following two-qubit partially entangled states:

$$|\psi\rangle_{AB} = \sqrt{p}|00\rangle_{AB} + \sqrt{1-p}|11\rangle_{AB} \quad (0 \leq p \leq 1). \quad (3)$$

(Hereafter, we omit the symbol of the tensor product.) Without loss of generality, we can focus on this form of entangled states in which no "cross terms" like $|01\rangle$ or $|10\rangle$ appear due to Schmidt decomposition. For simplicity, we discuss only two-qubit pure entangled states here, but the following argument also holds in general d-dimensional states as well, i.e., the situation in which Alice and Bob have d-level quantum systems instead of qubits. Then, the unique measure of bipartite entangled states is defined as follows [7]:

$$E(\psi) = -p\log_2 p - (1-p)\log_2(1-p). \quad (4)$$

This is von Neumann entropy of Alice's (or Bob's) reduced density matrix and also the same quantity as classical Shannon entropy. It is easily seen that it is maximum when the entangled state is a maximally entangled one, e.g., a Bell pair, in which the squared amplitudes have equally weighted probability distribution. Also, it takes the minimum value (zero) when the state is either $|00\rangle$ or $|11\rangle$, in accordance with our intuition, which implies that the states contain no entanglement. The amount of entanglement of partially entangled states lies between zero and one in the case of two-qubit states. Generally, d-level entangled states are quantified by d-level Shannon entropy, whose maximum is $\log_2 d$.

The measure has the following operational meaning. When we say that an entangled state $|\psi\rangle$ has entanglement $E(\psi)$, this means the state is equivalent to $E(\psi)$ copies of Bell pairs. This interpretation is based on two processes called entanglement distillation and entanglement dilution (or formation). Entanglement distillation enables us to extract maximally entangled states (Bell pairs) from a given state by only LOCC. In particular, imagine a situation in which Alice and Bob share n identical copies of a given state $|\psi\rangle$, and they try to distill as many Bell pairs as possible from them. They are allowed to manipulate the entire state of n copies collectively. Let m be the number of Bell pairs distilled and define $E_D(\psi) \equiv \lim_{n\to\infty} \frac{m_{\max}}{n}$. This quantity is called distillable entanglement and represents the maximum number of Bell pairs that can be distilled per copy. Conversely, we can imagine the opposite process, in which Alice and Bob try to form a given state from a supply of a large number of Bell pairs by only LOCC, which is entanglement dilution. Here again, suppose they share k copies of Bell pairs in advance and try to invest as few Bell pairs as possible to form a pretty good approximation to n copies of $|\psi\rangle$ by collective operations. Then, the quantity $E_C(\psi) \equiv \lim_{n\to\infty} \frac{k_{\min}}{n}$ is called the entanglement cost, which represents the minimum number of Bell pairs necessary to form the state per copy.

Quite surprisingly, in the case of pure states, these two quantities coincide. That is, we can distill as many Bell pairs as we invested in the formation per copy, and vice versa. Furthermore, it turns out that $E_D(\psi) = E_C(\psi) = E(\psi)$. This property is far from obvious because protocols under LOCC are highly nontrivial in general and not yet fully understood. These are among the very few examples of LOCC protocols that have been successfully established.

According to the above interpretation, we can always convert any bipartite pure state $|\psi\rangle$ into the other state $|\phi\rangle$ if and only if $E(\psi) \geq E(\phi)$ by the following procedure: First, Alice and Bob distill $E(\psi)$ copies of Bell pairs from $|\psi\rangle$ per copy, from which they construct $|\phi\rangle$ by LOCC. This is always possible because the number of Bell pairs necessary to construct $|\phi\rangle$ is smaller than the number that can be distilled from $|\psi\rangle$. Note that these operations should be performed asymptotically. Therefore, for any pair of bipartite pure states, one state can access the other, at least in one direction. This implies that the set of bipartite pure states is totally ordered.

This structure is exactly the same as that in thermodynamics mentioned in the previous section. Thermal equilibrium states and adiabatic processes in thermodynamics correspond to bipartite pure states and LOCC processes, respectively. In fact, with this analogy and a few other natural assumptions, we can derive the uniqueness of entropy of entanglement as an entanglement measure in a manner parallel to thermodynamics. This can be done by Giles's axiomatic approach to thermodynamics [4]. In summary, the totally ordered structure of bipartite pure states is essential for defining the unique measure, while we will see in the next section that partial order appears in mixed states.

5. Partial order on mixed entangled states

Contrary to our success in defining the unique measure of entanglement in bipartite pure states, it turns out that in this section we fail in our attempt to find a unique measure for mixed-state entanglement as long as we stick to the LOCC framework. That is, in the most general class of entangled states, accessibility between two arbitrary states does not hold any longer, which implies that it is impossible to establish a physically reasonable unique measure of entanglement, at least under LOCC.

In the following, it is shown that there exist two mixed entangled states, ρ_{AB} and σ_{AB}, that are not accessible in either direction, i.e., $\rho_{AB} \nrightarrow \sigma_{AB}$ and $\sigma_{AB} \nrightarrow \rho_{AB}$. This is done by employing a very peculiar entangled state called a bound entangled state [8]. Bound entangled states are intrinsic to the mixed-state regime and do not appear in pure entangled states. They are peculiar in that one cannot distill any pure-state entanglement from them, while one must invest some amount of entanglement to construct them by LOCC. Thus, they have a sort of irreversibility in terms of the entanglement they consume and produce.

The definitions of distillable entanglement and entanglement cost given in the previous section are naturally applicable to mixed states as well. Distillable entanglement represents the number of Bell pairs that can be extracted from a given mixed entangled state ρ_{AB} per copy. This includes more practical meaning than pure-state cases because mixed entangled states could appear due to decoherence processes during the transmission of half a pure entangled pair from Alice to Bob to establish quantum correlation between them prior to teleportation. Thus, distillable entanglement quantifies the amount of entanglement one can recover from unwanted decoherence processes, although we will not go into details of entanglement distillation protocols here. According to the definition of entanglement cost for pure states, it represents the number of Bell pairs one needs to invest to form a given mixed entangled state. However, a big difference from pure-state cases is that here Alice and Bob need to discard some information they used in the formation process into the environment in order to induce mixed states. Due to this process of information loss, a sort of irreversibility enters into mixed entangled states, and one cannot recover all the entanglement invested in the formation phase. This greatly contrasts with the case of pure states where distillable entanglement and entanglement cost coincide.

In terms of these quantities, a bound entangled state ρ_{AB} is a state for which $E_D(\rho_{AB}) = 0$. However, it is natural to expect that even if ρ_{AB} is undistillable, it should need some amount of entanglement when it is created by LOCC, i.e., $E_C(\rho_{AB}) > 0$. In fact, it has been found that such bound entangled states exist [9], i.e., $E_D(\rho_{AB}) = 0$ but $E_C(\rho_{AB}) > 0$, based on which we will next construct an example of a pair of states that are not accessible from each other.

Before moving on to the partial order argument, let us define the notion of accessibility in our context. We say a state ϕ is accessible from a state ψ when there exists an asymptotic LOCC protocol such that n copies of ψ can be converted into an arbitrarily good approximation of n copies of ϕ. Note that we require the accessibility between the same number of copies here. For the detailed description of this definition and the following argument, see Ref. [10].

Suppose we take a pure entangled state $|\phi\rangle_{AB}$ such that

$$0 < E_C(\phi) < E_C(\rho). \tag{5}$$

Then, by definition, the bound entangled states ρ_{AB} cannot be converted into the pure state $|\phi\rangle_{AB}$. Thus, $|\phi\rangle_{AB}$ is not accessible from ρ_{AB} under LOCC, i.e., $\rho_{AB} \not\rightarrow |\phi\rangle_{AB}\langle\phi|$. On the other hand, by using the property of entanglement cost E_C that it cannot be increased by LOCC, it is shown that ρ_{AB} is also not accessible from $|\phi\rangle_{AB}$. Roughly speaking, this can be understood as follows: entanglement cost represents the minimum number of Bell pairs necessary to form a given entangled state by LOCC. Thus, if it were possible to convert an entangled state with a lower entanglement cost into another state with a higher entanglement cost by LOCC, there would be a contradiction because the latter state would have to be constructed from a smaller number of Bell pairs than its entanglement cost, which is an apparent contradiction of the definition of entanglement cost. Therefore, we have just shown that ρ_{AB} and $|\phi\rangle_{AB}$ are not accessible in either direction under LOCC. This implies that the set of mixed entangled states is partially ordered (**Fig. 2**). As we saw in Sec. 3, partial order is the sufficient condition for the non-uniqueness of a measure. Therefore, we cannot expect a unique measure of entanglement for mixed states under LOCC.

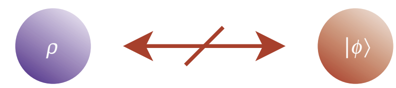

Fig. 2. Partial order on mixed entangled states. A bound entangled state ρ_{AB} and a bipartite pure state $|\phi\rangle_{AB}$ are chosen such that $0 < E_C(\phi) < E_C(\rho)$. They are not accessible from each other in either direction.

What we have shown reveals that the non-uniqueness of entanglement measure originates in the partial order structure of the set of mixed entangled states. This clearly contrasts mixed-state entanglement with pure-state entanglement, where we can draw a very beautiful analogy with classical thermodynamics. The fact that we need to choose different measures for mixed-state entanglement in different scenarios reflects that the set of mixed entangled states changes its appearance when viewed from different "angles" due to the partial order.

6. Summary

In this paper, I reviewed the quantification of entanglement from an operational point of view based on a very general notion of accessibility between physical states. To be quantified uniquely under LOCC, the set of entangled states should be totally ordered under LOCC. When we restrict ourselves to bipartite pure states, this certainly holds and a unique measure, the entropy of entanglement, can be defined. However, the set of mixed entangled states has a partially ordered structure, so it is impossible to find a unique measure under LOCC. This suggests that, unlike other simple resources in physics, quantum entanglement is a highly complicated physical resource, which implies that it might be potentially far more powerful and have a richer underlying structure than presently expected. The theory of entanglement is still in its infancy and we have just started to recognize its potential as a new type of physical resource. A deeper understanding of the nature of entanglement will shed some light on the power of quantum information processing.

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References

[1] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, "Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels," Phys. Rev. Lett., Vol. 70, p. 1895, 1993.
[2] A. K. Ekert, "Quantum Cryptography Based on Bell's Theorem," Phys. Rev. Lett., Vol. 67, p. 661, 1991.
[3] R. P. Feynman, "Quantum Mechanical Computer," Opt. News, Vol. 11, p. 11, 1985; D. Deutsch, "Quantum Theory, the Church-Turing Principle and the Universal Quantum Computer," Proc. R. Soc. Lond., Vol. A 400, p. 97, 1985.
[4] V. Vedral and E. Kashefi, "Uniqueness of Entanglement Measure for Bipartite Pure States and Thermodynamics," Phys. Rev. Lett., Vol. 89, p. 037903, 2002.
[5] S. Virmani and M. B. Plenio, "Ordering States with Entanglement Measures," Phys. Lett., Vol. A 268, p. 31, 2000.
[6] M. A. Nielsen and I. L. Chuang, "Quantum Computation and Quantum Information," Cambridge University Press, Cambridge, 2000.
[7] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, "Concentrating Partial Entanglement by Local Operations," Phys. Rev., Vol. A 53, p. 2046, 1996; S. Popescu and D. Rohrlich, "Thermodynamics and the Measure of Entanglement," *ibid*, Rev., Vol. A 56, p. R3319, 1997.
[8] M. Horodecki, P. Horodecki, and R. Horodecki, "Mixed-state Entanglement and Distillation: Is there a "Bound" Entanglement in Nature?" Phys. Rev. Lett., Vol. 80, p. 5239, 1998.
[9] G. Vidal and J. I. Cirac, "Irreversibility in Asymptotic Manipulations of Entanglement," Phys. Rev. Lett., Vol. 86, p. 5803, 2001.
[10] F. Morikoshi, M. F. Santos, and V. Vedral, "Accessibility of physical states and non-uniqueness of entanglement measure," J. Phys. A: Math and Gen., Vol. 37, p. 5887, 2004.

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