Representation Theory and Combinatorics Arising from Determinants

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Abstract

Originating from the unified treatment of probability distributions, the α-determinant is a one-parameter interpolation of the determinant and permanent of a matrix. While it generally does not have the invariance properties of the determinant, the irreducible representations of the associated general linear (Lie) groups define interesting invariants that unveil a new and rich mathematical theory with abundant unsolved problems, related to symmetric functions, representation theory, combinatorics, number theory, probability and other areas.

Keywords: α-determinant, wreath determinant, irreducible decomposition of representations

1. Introduction

In this article, we focus on matrix Lie groups, that is, groups of matrices with the operation of the usual matrix product (i.e., composition of linear forms), the most basic example of which is the general linear group over the complex numbers. Since Lie groups are also geometrical objects (differentiable manifolds), both the group structure and differentiable structure coexist in a single object and must be accounted for. Starting from certain special representations of these groups, we introduce research on several areas of mathematics, including new research directions and open problems.

1.1 Groups

A group G is a set with an operation (generically called "product") $G \times G \ni (g, h) \to gh \in G$ that is associative, i.e., $(gh)k = g(hk)$ for g, h, $k \in G$. In addition, G must have an identity element $e \in G$ such that $ge = eg = g$ holds for all $g \in G$, and for all $g \in G$ there must be an inverse element g^{-1} such that $gg^{-1} = g^{-1}g = e$.

The main examples of groups in this article are the symmetric group of m elements $\mathfrak{S}_m$, i.e., the groups of permutations of m letters with the operation of composition, and the general linear group $GL_n(\mathbb{R})$ (resp. $GL_n(\mathbb{C})$), the group of n dimensional square nonsingular (invertible) matrices with real (resp. complex) entries with the usual matrix multiplication.

1.2 Lie groups, Lie Algebras and their representations

A representation (π, V) of G is a group homomorphism π from G to the group of linear transformations $GL(V)$ of a vector space V. We normally consider only vector spaces over the complex numbers. In this setting, we say that G acts on V, or that V is a G-module, and if it is clear from the context, we omit the notation π. To consider an infinite dimensional V, it is indispensable to introduce the notion of topology; thus, we only consider finite dimension. By introducing a Hermitian inner product to the n-dimensional V, we may consider the group $U(n)$ of unitary matrices, the group of matrices that leave the inner product invariant. The development of representation theory took place in parallel with the revolutionary physical theories of relativity and quantum physics, resulting in a notation and terminology that mixes mathematical and physical subtleties and may be confusing at first but that is not without merit.

As mentioned above, Lie groups are manifolds with

geometric structure and notions such as "curvature" and "connectedness," so they cannot be described faithfully only with linear algebra. Because of this, and to the extent possible, instead of G, we consider the associated "differentiated" Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ is equivalent as a vector space to the set of tangent vectors at the identity $e \in G$, its action is regarded as the infinitesimal version of the Lie group action. In this article, we only consider finite-dimensional representations, so Lie algebras suffice, but we note that this is not the case for general Lie groups and representations.

The Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ of the Lie group $GL_n(\mathbb{C})$ is the ring of all n-dimensional matrices. For a matrix $X \in \mathfrak{gl}_n(\mathbb{C})$, the exponential map satisfies $\det(\exp X) = e^{\mathrm{tr}X} \neq 0$, thus $\exp X \in GL_n(\mathbb{C})$ is the corresponding Lie group element. Therefore, if the action of $GL_n(\mathbb{C})$ on a polynomial f over Mat_n is $(g.f)(x) = f(g^{-1}x)$, the infinitesimal action of $\mathfrak{gl}_n(\mathbb{C})$ is given by

$$(X.f)(x) = \frac{d}{dt} f(\exp(-tX)x)|_{t=0}.$$

1.3 Motivation for the representation theoretical study of α-determinant

The α-determinant is a generalization of both the determinant and permanent*. In the definition of the α-determinant, given below, the sign changes appearing in the definition of the determinant are replaced with certain weights depending on the parameter α. By setting $\alpha = -1$ we recover the definition of the usual determinant and with $\alpha = 1$, that of the permanent. It was originally introduced as α-permanent by Vere-Jones [1] in the context of probability theory. The name α-determinant follows the convention of Tomoyuki Shirai and Yoichiro Takahashi [2]. They used the α-determinant to construct point processes that generalize the ones used in finance and time-series analysis, namely, the boson point process, fermion point process, and Poisson point process.

Surprisingly, when $\alpha \neq -1$ the α-determinant loses the multiplicative property of the usual determinant, that is, $\det(AB) = \det(A)\det(B)$. Thus, we immediately ask the following problem question.

Problem 1: Where did the multiplicative property go?

The determinant is a one-dimensional representation of $GL_n(\mathbb{C})$. While the permanent does not define a one-dimensional representation, it defines an irreducible representation of the vector space spanned by the symmetric tensor product. With irreducible repre-

sentation, we refer to a subspace of V, different from $\{0\}$ and V itself, that is invariant under the action of G. Informally, we may think of irreducible representations as prime numbers or elementary particles in particle physics. Therefore, we may also ask the following problem question.

Problem 2: For general α, what is the structure of the spaces spanned as the group acts on the α-determinant?

These two questions sparked the interest of the second author of this article in the representation theory of the α-determinant [3].

2. The α-determinant

For a fixed α, the α-determinant of a square matrix $A = (a_{ij}) \in \mathrm{Mat}_n$ is a modification of the usual determinant defined as

$$\det^{(\alpha)}A := \sum_{\sigma \in \mathfrak{S}_n} \alpha^{n-\nu_n(\sigma)} a_{\sigma(1)1} \cdots a_{\sigma(n)n}.$$

Here, $\nu_n(\sigma)$ is the cycle number of a permutation $\sigma \in \mathfrak{S}_n$, i.e., the number of cycles in the cycle decomposition of σ. Hereafter, we write $\nu(\sigma) := n - \nu_n(\sigma)$ and note that $\nu(\sigma)$ is the minimum number of transpositions required to express σ.

A theorem of Vere-Jones [1] is that $\det(I_n - \alpha TA)^{-\frac{1}{\alpha}}$ can be expanded in terms of the α-determinant as

$$\det(I_n - \alpha TA)^{-\frac{1}{\alpha}}$$

$$= \sum_{k \geq 0} \frac{1}{k!} \sum_{i_1,\ldots,i_k \geq 0} t_{i1} \cdots t_{ik} \det^{(\alpha)}\left(a_{i_p,i_q}\right)_{1 \leq p,q \leq k},$$
$$T = (t_{ij}) \in \mathrm{Mat}_n.$$

We refer the reader to a previous study [4] for details on the discovery of the formula and its applications.

2.1 Preliminaries from representation theory

It is well known that the finite dimensional irreducible representations (irreducible $\mathfrak{gl}_n$-modules) of $GL_n(\mathbb{C})$ and its Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ are parametrized by the so-called highest weights. With respect to the standard action of $GL_n(\mathbb{C})$ on $\mathbb{C}^n$, the natural action of $GL_n(\mathbb{C})$ on the tensor product $(\mathbb{C}^n)^{\otimes m}$ commutes with the action of $\mathfrak{S}_m$ permuting the m spaces in the tensor product, because there is a duality between the two actions. This is the well-known Schur–Weyl duality [5]. From this perspective, the highest weights are

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* Permanent: A matrix function like the determinant but with no sign changes in the definition.
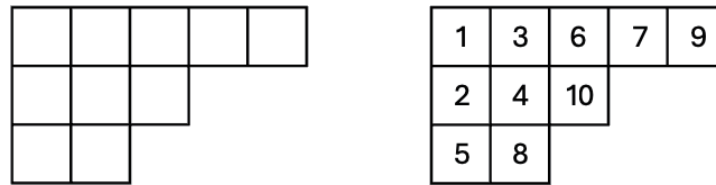
(a) Young diagram of the partition $(5,3,2) \vdash 10$ (b) Example of standard tableau

Fig. 1. Young diagrams: visual representations of partitions.

identified with partitions of integers. We say that the vector of non-negative integers $\lambda = (\lambda_1, \lambda_2, ..., ..)$ is a partition of n, written as $\lambda \vdash n$, when it satisfies $n = \lambda_1 + \lambda_2, + \cdots (\lambda_1 \geq \lambda_2 \geq \cdots)$. The length $l(\lambda)$ is the number non-zero parts of the partition λ.

A λ is often identified with the associated Young diagram, defined by

$$\{(i, j) \in \mathbb{Z}^2 \mid 1 \leq j \leq \lambda_i, i \geq 1\}.$$

It is customary to illustrate the Young diagrams as left-aligned arrangements of boxes with λ_i boxes in the i-th row, as shown in **Fig. 1(a)**. If we arrange the n numbers 1, 2, ..., n in a Young diagram λ such that numbers in a row increase from left to right and in each column the numbers increase from top to bottom we obtain a standard tableau of shape λ (**Fig. 1(b)**). The set $\mathrm{Stab}(\lambda)$ is the set of all standard tableaus of shape λ and is a fundamental notion in the representation theory of the symmetric group.

2.2 Cyclic modules arising from α-determinant

The action of $\mathfrak{gl}_n(\mathbb{C})$ on the ring of polynomial functions $\mathcal{A}(\mathrm{Mat}_n)$ on the variables $\{x_{ij}\}_{1 \leq i, j \leq n}$ is given by the differential operators

$$E_{ij}.f = \sum_{k=1}^{n} x_{ik} \frac{\partial f}{\partial x_{jk}}.$$

Here, $\{E_{ij}\}$ $(1 \leq i, j \leq n)$ is the standard (matrix elements) basis of $\mathfrak{gl}_n(\mathbb{C})$.

For $X = (x_{ij})$, we denote as $V_n(\alpha)$ the cyclic $\mathfrak{gl}_n(\mathbb{C})$-module generated by $\det^{(\alpha)}X$. In other words, $V_n(\alpha)$ is the vector space (module) generated by the (repeated) action of elements of $\mathfrak{gl}_n(\mathbb{C})$ on $\det^{(\alpha)}X \in \mathcal{A}(\mathrm{Mat}_n)$. For $\alpha = -1$, we have $\det^{(\alpha)}X = \det X$; thus, $V_n(-1) = \mathbb{C} \cdot \det X$. Therefore, the study of the structure of $V_n(\alpha)$ is an important step towards the answer to both **Problems 1** and **2**.

Theorem 2.1 [3]. Let $\mathbf{E}_n^\lambda$ be an irreducible $\mathfrak{gl}_n(\mathbb{C})$-

module of highest weight λ. Then, $V_n(\alpha)$ has the irreducible decomposition

$$V_n(\alpha) \cong \bigoplus_{\substack{\lambda \vdash n \\ f_\lambda(\alpha) \neq 0}} (\mathbf{E}_n^\lambda)^{\oplus f^\lambda}.$$

Here, $f^\lambda := |\mathrm{Stab}(\lambda)|$ and $f_\lambda(x) := \prod_{(i,j) \in \lambda} (1 + (j - i)x)$ is the content polynomial of λ.

This theorem shows that if $\alpha \notin \{\pm 1, \pm \frac{1}{2}, ..., \pm \frac{1}{n-1}\}$, $V_n(\alpha)$ is equivalent to the standard representation of the tensor product $(\mathbb{C}^n)^{\otimes n}$ of $\mathbb{C}^n$, and that the structure degenerates when α is the reciprocal of an integer with an absolute value smaller than n.

It is also natural to generalize this situation and consider the cyclic $\mathfrak{gl}_n$-module generated by the power $(\det^{(\alpha)}X)^l$. In this case, the highest weight modules appearing in the decomposition have nontrivial multiplicity given by a combinatorial quantity, and computing this multiplicity is generally a difficult problem. When $n = 2$, the pair $(\mathfrak{S}_{2l}, \mathfrak{S}_l \times \mathfrak{S}_l)$ of a symmetric group and Young subgroup is a Gelfand pair, which enables one to write the multiplicity in terms of zonal spherical functions given explicitly by hypergeometric polynomials [6]. The general case is still open and currently there is not even a conjectural form for the multiplicity. Moreover, one of the classic and fundamental notions in the theory of symmetric functions is plethysms [7]. In a previous study [8], for the power of $\det^{(1)}(X)$ (permanent) and the generated cyclic module structure, a conjecture that is related with plethysms for the symmetric tensor product was presented. The case of general n appears to be difficult, but it has been verified for $n = 2$ and arbitrary l, as well as for $n = 3$ and $l = 2$. Note that analogous research has been conducted for quantum groups [9]. For $n = 2$, the multiplicity is given both by hypergeometric polynomials and q-hypergeometric polynomials. This is a fascinating and promising line of research [10, 11].

3. Wreath determinant

As discussed in the previous section, when α takes the values $\pm\frac{1}{k}$, the structure of the cyclic module structure arising from the α-determinant changes drastically. A careful analysis shows that when $\alpha = -\frac{1}{k}$, there is a weak form of the alternating property of the usual determinant.

Let K denote the subgroup of $\mathfrak{S}_n$ consisting of permutations that fix any elements except from $k + 1$ taken (arbitrary) from 1, 2, ..., n. Then, for $A \in \mathrm{Mat}_n$ we observe that the sum $\sum_{\sigma \in K} \det^{(\alpha)}(AP(\sigma))$ contains the factor $(1 + \alpha)(1 + 2\alpha) \ldots (1 + k\alpha)$ ($A \in \mathrm{Mat}_n$), where $P(\sigma)$ is the permutation matrix of $\sigma \in \mathfrak{S}_p$. In other words, the sum $\sum_{\sigma \in K} \det^{(-\frac{1}{k})}(AP(\sigma))$ vanishes and if $k + 1$ columns of A are equal we have $\det^{(-\frac{1}{k})}A = 0$, generalizing the alternating property of the usual determinant.

Since $\det^{(\alpha)}(AP(\sigma)) = \det^{(\alpha)}(P(\sigma)A)$, the foregoing discussion holds similarly for rows. A consequence of the weak alternating property is that we have analogs of Vandermonde and Cauchy determinant formulas [12].

3.1 Invariant theory of wreath determinants

We denote by $1_{p,q}$ the matrix of size $p \times q$ with all entries equal to 1. For $A \in \mathrm{Mat}_{n,kn}$, the k-wreath determinant is defined by

$$\mathrm{wrdet}_k A := \det^{(-\frac{1}{k})}(A \otimes 1_{k,1}).$$

For example, we have

$$\mathrm{wrdet}_2 \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

$$= \det^{(-\frac{1}{2})} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

$$= \frac{a_1 a_2 b_3 b_4}{4} - \frac{a_1 a_3 b_2 b_4}{8} - \frac{a_2 a_3 b_1 b_4}{8}$$
$$- \frac{a_1 a_4 b_2 b_3}{8} - \frac{a_2 a_4 b_1 b_3}{8} + \frac{a_3 a_4 b_1 b_2}{4}.$$

Like the usual determinant, the wreath determinant is characterized by its left-GL_k, right-$\mathfrak{S}_k \wr \mathfrak{S}_n$ relative invariance. Here, $\mathfrak{S}_k \wr \mathfrak{S}_n$ is the wreath product of the groups $\mathfrak{S}_k$ and $\mathfrak{S}_n$ and is defined using the concept of the semidirect product of groups (a technique used to obtain a new group from two known groups).

Theorem 3.1 [12]. A map f: $\mathrm{Mat}_{n,kn} \to \mathbb{C}$ with properties

1. f is a multilinear map with respect to the columns,

2. $f(QA) = (\det Q)^k f(A)$ holds for any $Q \in \mathrm{Mat}_n$, that is, f is left-GL_k relative invariant, and
3. $f(AP(\sigma)) = \pm f(A)$ holds for any $\sigma \in \mathfrak{S}_k \wr \mathfrak{S}_n$ (if $\sigma \in \mathfrak{S}_k^n$ we have $f(AP(\sigma)) = f(A)$)

is equal to the map $A \mapsto \mathrm{wrdet}_k A$ up to a scalar multiple.

While this theorem may be proved in an elementary manner, it is also an easy consequence of the invariant theory of the underlying (GL_n, GL_{kn})-duality.

3.2 An analog of group determinant for group-subgroup pairs using wreath determinants

Ferdinand Georg Frobenius developed the theory of group characters in his study of the group determinant for non-abelian groups. For a finite G, consider an indeterminate x_g for each group element $g \in G$, then the group determinant is given by

$$\Theta(G) := \det(x_{uv^{-1}})_{u,v \in G}.$$

The group determinant is a complete invariant with respect to the isomorphism classes of groups. Concretely, it holds that

$$\Theta(G) = \Theta(G') \Leftrightarrow G \cong G'.$$

The main result of Frobenius is the factorization of the group determinant into irreducible polynomials with coefficients given by the group characters, in modern terms this is equivalent to the decomposition of the regular representation of a group (the action of G on its group ring) into irreducible representations.

The origin of the work of Frobenius in the group determinant is a letter by Richard Dedekind in 1896. Dedekind had previously discovered how to factorize the group determinant for abelian groups and posed the problem for general non-abelian finite groups to Frobenius. It is not an exaggeration to say that this letter marked the beginning of representation theory of finite groups.

Recall that the k-wreath determinant is defined for $n \times kn$ matrices; therefore, by considering a pair (G, H) of a finite G and subgroup H of index k, by analogy we may define

$$\Theta(G, H) := \mathrm{wrdet}_k(x_{hg^{-1}})_{h \in H, g \in G}.$$

Note that since the wreath determinant does not have an invariance with respect to column transpositions, this definition depends on the ordering of the matrix columns and rows. Therefore, it is necessary to consider a fixed numbering (or naming) of the group elements. For a fixed bijection ϕ: $\{0, 1, ..., kn - 1\} \to G$, we say that $g_i = \phi(i)$ is a numbering of G. For a

$\mathbb{C}$-algebra A (e.g. a polynomial ring in $kn - 1$ variables) a map $f\colon g_i \mapsto q^i$ is called specialization. When G is an abelian group, for a reasonable numbering of the elements of G, we may consider the specialization $f\colon g_i \mapsto q^i$ to see that the determinant $\Theta(G, H)$ factors into products of polynomials of the form $q^i - 1$ [13].

Let us give a concrete example. From the partition $(k, \dots, k) \vdash kn$ and corresponding irreducible characters $\chi^{(k^n)}$ of $\mathfrak{S}_k^n$, we define a $\mathfrak{S}_k^n$-bi-invariant function $\omega^{(k^n)}$ on $\mathfrak{S}_{kn}$ by averaging over $\mathfrak{S}_k^n$. Concretely, we define

$$\omega^{(k^n)}(x) = \frac{1}{(k!)^n} \sum_{g \in \mathfrak{S}_k^n} \chi^{(k^n)}(xg) \quad (x \in \mathfrak{S}_{kn}).$$

Example 3.2. For $G = \mathbb{Z}_n \times \mathbb{Z}_n$ and $H = \mathbb{Z}_n \times \{0\}$, we have

$$\Theta(G, H) = \omega^{(n^n)}(\sigma\tau^{-1}) \left(\frac{n!}{n^n}\right)^n q^{\frac{n^3(n-1)}{2}}$$

$$(q^n - 1)^{n(n-1)}.$$

Here, $\sigma, \tau \in \mathfrak{S}_{n^2}$ and we define $P(\tau) = P((1\ 2\ \dots\ n)) \otimes P((1\ 2\ \dots\ n))$, $1_{1,n} \otimes I_n = (I_n \otimes 1_{1,n})P(\sigma)$ to obtain

$$\omega^{(n^n)}(\sigma\tau^{-1}) = \frac{1}{n!^n} \times \left\{ \text{coefficient of square-}\right.$$
$$\left. \text{free elements} \prod_{i,j=1}^n x_{ij} \text{ in } \left(\det(x_{ij})_{1 \le i,j \le n}\right)^n \right\}.$$

Note that if $n = 3$, we have $\Theta(G, H) = 0$.

A comprehensive theory of group-subgroup wreath determinant has not yet been developed, but two related developments have surfaced, one is related to the Alon–Tarsi conjecture by Kazufumi Kimoto and the other is the construction of new graphs based on group and subgroups.

3.3 Alon–Tarsi conjecture

A Latin square is a square arrangement (i.e. a $n \times n$ matrix) filled with the numbers $1 \sim n$ in such a way that in each row and column, each number appears exactly once. For instance, the 12 Latin squares of size 3 are

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

The sign sgn L of a Latin square L is the product of the signs of all column and row permutations. If sgn $L = 1$, we say L is even and odd if sgn $L = -1$.

If n is an odd number, the formula $\text{sgn}(P((1,2))L) = (-1)^n \text{sgn } L$ holds for the transposition $(1,2)$, giving a bijection between odd and even Latin squares and showing that the number of odd and even Latin squares of odd size n is equal. For even numbers the problem is still open.

Conjecture 3.3 (Alon–Tarsi conjecture 1992). For an even number n, the number of even and odd Latin squares of size n is different.

While the conjecture originated in graph coloring problems, it has interesting and nontrivial equivalent formulations such as the Rota basis conjecture on a certain special choice of basis for an n dimensional vector spaces, and the non-vanishing of certain integrals over the special unitary group $SU(n)$ with respect to the square-free coordinate product Haar measure ($SU(n)$ bi-invariant measure). Kimoto also discovered the equivalence of the Alon–Tarsi conjecture in terms of the wreath determinant [12].

Theorem 3.4. Let n be an even number. The Alon–Tarsi conjecture is equivalent to any of the following three statements below.

1. $\text{wrdet}_n \left(\overbrace{I_n\ I_n\ \dots\ I_n}^{n}\right) \neq 0$,

2. $\omega^{(n^n)}(g_n) \neq 0$,

3. $\sum_{y \in \mathfrak{S}_n^n} \left(-\frac{1}{n}\right)^{\nu(g_n y)} \neq 0$.

Here, $g_n \in \mathfrak{S}_{n^2}$ is a permutation satisfying $g_n((i-1)n + j) = (j-1)n + i (1 \le i,j \le n)$ and $\mathfrak{S}_n^n$ is the set of permutations of $\mathfrak{S}_{n^2}$ that fix the set $\{(i-1)n + j | 1 \le j \le n\}$ $(i = 1, \dots, n)$.

The Alon–Tarsi conjecture is easy to verify for specific cases, but as usual in number theory and combinatorics, the proof appears to be a formidable challenge. The best results to date are those of Drisko (1997) for $n = p + 1$, where p is a prime number, and of Glynn (2010) for $n = p - 1$ (see [14] for a review up to 2019). It is worth mentioning that an alternative proof of the latter result using the wreath determinant has been given by Kimoto [15, 16].

For the number of Latin squares $\text{ls}(n)$, although it is a weak result, we remark that the asymptotic formula $\text{ls}(n)^{1/n^2} \sim e^{-2}n$ is known [17]. One of the reasons that explains that the Alon–Tarsi conjecture may be difficult to solve is that the number of even and odd Latin squares appear to be asymptotically equal.
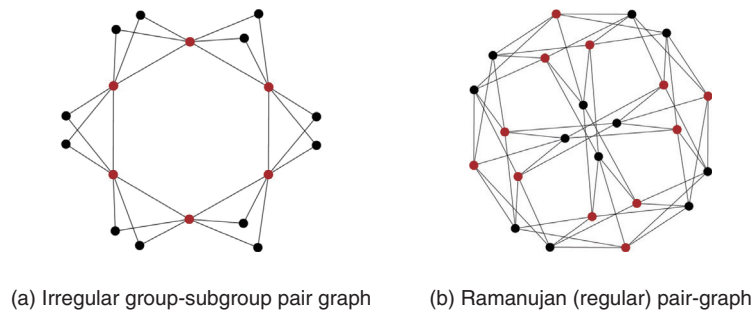
(a) Irregular group-subgroup pair graph   (b) Ramanujan (regular) pair-graph

Fig. 2.   (a) Pair-graph of cyclic group of 18 elements, cyclic group of 6 elements. (b) Pair-graph of symmetric group $\mathfrak{S}_4$, alternating group A_4.

Let us propose a new research problem.

Problem 3.5. Find a suitable definition of a Latin square zeta function $\zeta_{LS}(s)$ such that we have the following equivalence:
   best estimate in the prime number theorem ($\leftrightarrow$ Riemann hypothesis): $\zeta(s)$ = Alon–Tarsi conjecture $\zeta_{LS}(s)$.

In other words, such that the Alon–Tarsi conjecture is equivalent to a Riemann hypothesis' analog for $\zeta_{LS}(s)$, it might be helpful to compare the Alon–Tarsi conjecture with the phenomenon of Chebyshev's bias for prime numbers.

3.4 Group-subgroup pair graphs

In graph theory, there is an important class of graphs known as Ramanujan graphs. Ramanujan graphs are theoretically the best expanders, i.e., graphs with rapid mixing and good diffusion properties. The properties of Ramanujan graphs make them particularly important for applications, including the construction of cryptographic hash functions. A graph being Ramanujan is equivalent to the analog of Riemann Hypothesis for the Ihara zeta function of the graph, making it also interesting from the point of view of number theory.

The main problem in this area is the construction of explicit families of Ramanujan graphs of fixed degree. Some known examples are the Lubotzky–Phillips–Sarnak graphs (1994), constructed with Cayley graphs of projective linear groups over finite fields using the theory of Hamiltonian quaternion algebras and their maximal orders, and Pizer graphs (1990), based on the theory of isomorphism classes (isogenies) of hyperelliptic curves over finite fields. In the former, for a prime number p congruent to 1

modulo 4, the resulting graphs are a family of (p + 1)-regular Ramanujan graphs. A Cayley graph is a regular graph constructed from the elements of a finitely generated G and where the edges are defined using the (group) multiplication using a generating set S.

A new research direction is defining hash functions for group-subgroup pair graphs, which are a generalization of Cayley graphs (see **Fig. 2**), and studying their cryptographic properties [18]. For a G, subgroup H and a subset $S \subset G$ such that $S \cap H$ is a symmetric set, the group-subgroup pair graph, or simply pair-graph, $\mathcal{G}(G, H, S)$ is a graph defined in a manner analogous to Cayley graphs, but such that the resulting graph is not generally regular. Concretely, $\mathcal{G}(G, H, S)$ is a graph, the vertices of which are the elements of G and such that $h \in H$ and $g \in G$ are connected by an edge when they satisfy the relation $h \sim g \Leftrightarrow g = hs$ ($s \in S$) [19]. While examples of Ramanujan group-subgroup pair graphs are known for the regular case (see Fig. 2(b)), infinite families have not yet been constructed. The construction of infinite families of Ramanujan graphs is one of the major goals in the theory of pair-graphs from the point of both mathematics and applications

4. Perspectives on positive singular values: Wishart distribution, Wallace sets, and positivity of α-determinant

A natural problem is to consider the positive case $\alpha = \frac{1}{k}$ as a generalization of the permanent in a similar manner to the wreath determinant for negative singular values. We conclude this article with hints that may serve as a starting point for this research.

The study of positive definite functions on convex cones on Euclidean spaces has applications to several

areas, including optimization and probability. In the analysis of Hilbert spaces of holomorphic functions on symmetric cones (symmetric spaces of Lie groups) and its unitary representation, there is an important notion called Wallach set. For the integral kernel of symmetric cones, the set is basically given by reciprocals $\alpha = \frac{1}{k}$ of the positive singular values [20].

In statistics, there is multivariate generalization of the χ-square distribution for positive definite matrices, called Wishart distribution. Let n mutually independent p-variate random vectors $\{x_1, x_2, ..., x_n\}$, with $n \geq p$, follow a multivariate normal distribution $N(0, \Sigma)$ with mean 0 and covariance matrix Σ. Then, the random variable $X = \sum_{k=1}^{n} x_i {}^t x_i$ is said to follow the Wishart distribution. The Wishart distribution is used to model the distribution of a sample covariance matrix for data following a multivariate normal distribution, after scaling by sample size. The Wishart distribution, due to the intimate relation with symmetric cones of positive definite matrices, also has applications to the representation theory of Jordan algebras [20]. On the other hand, it is known that the α-determinant has an interpretation in terms of the Wishart distribution [21]. Using this relation, Shirai proved that if α is an element of $\{2/k\}_{1 \leq k < n}$, or $\{-1/k\}_{1 \leq k < n}$, then the α-determinant is non-negative for non-negative definite Hermitian real symmetric matrices. The proof uses Jack polynomials [7], an important example of symmetric functions. As is evident, these values also correspond to the positive singular values described in this article and the values of the Wallach set. It is difficult to expect that all of this is a coincidence, so an important research direction is to clarify this situation and explain the relation between these distinct areas.

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